Resting a disk on a rough incline

In a recent paper,[1] De Luca works out how a thin disk can be rested on a rough incline by adhering an additional mass at its rim. De Luca's analysis is correct and rigorous, but I think it may be difficult for most high school students, and the physics of the solution(s) need to be more elaborated. Here is an alternative, and comparatively simpler approach.

Denote the center, radius, and mass of the disk as C, R, and M, respectively, the additional mass as m, and the sloping angle of the incline as θ. The center of mass (CM) of the composite disk-and-m object is at a distance r from C on the line connecting C and m, where $r = mR/(M+m)$.[2] Hence,

$$R/r = 1+\rho, \quad (1)$$

where $\rho = M/m$. It is obvious that for achieving a static equilibrium, CM *must be positioned exactly above* O, the contact point of the disk and the incline. Since the normal force N, and the friction fr act on the disk at O, the third force, $W = (M+m)g$, must point in a direction passing through O as well;

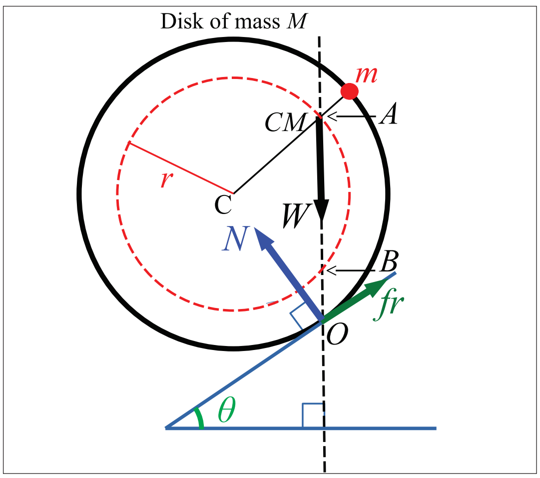

Fig. 1. Only when the mass m is placed such that CM, the center of mass of the disk-and-m object, is directly above O, an equilibrium can be achieved.

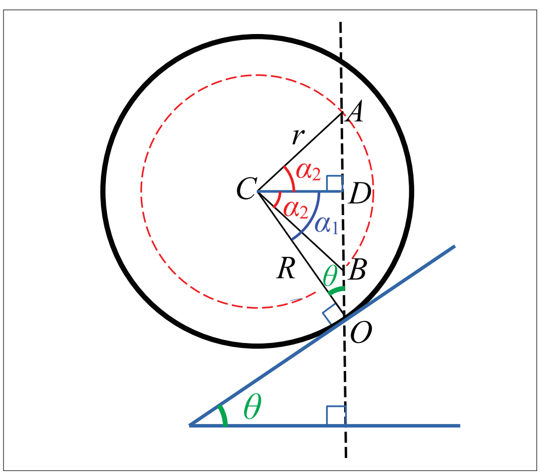

Fig. 2. By finding α_1 and α_2, the angular positions of A and B are determined.

otherwise, the net torque w.r.t. O will not be zero and thus the disk will rotate about O. As shown in Fig. 1, there are at most two such positions CM can situate (labeled as A and B), agreeing with the two distinct solutions found in Ref. 1. However, if the vertical line at O does not intersect the circle of radius r (r too small or θ too large), the disk cannot be set at equilibrium in any way.

Referring to the symbols in Fig. 2, the following results are straightforward. In $\triangle$COD,

$$\alpha_1 = \pi/2 - \theta, \quad (2)$$

and

$CD = R \sin\theta$. In $\triangle$CBD or $\triangle$CAD, $\cos\alpha_2 = CD/r = (R/r)\sin\theta$,

or

$$\cos\alpha_2 = (1+\rho)\sin\theta. \quad (3)$$

Thus, the problem is solved.

The angles of A and B, measured from CO, are $\alpha = \alpha_1 + \alpha_2$ and $\alpha = \alpha_1 - \alpha_2$, respectively. With the identity $\cos\alpha = \cos(\alpha_1 \pm \alpha_2) = \cos\alpha_1 \cos\alpha_2 \mp \sin\alpha_1 \sin\alpha_2$, the exact Eq. (15) of Ref. 1 is derived if $\sin\theta$ and $\cos\theta$ are expressed in terms of $t = \tan\theta$. The equilibriums A and B exist only when the angle α_2 exists, i.e., the vertical line at O cuts, or at least touches, the circle of radius r, implying the satisfaction of the condition

$$\sin\theta \le 1/(1+\rho), \quad (4)$$

corresponding to the right-half of Eq. (13) of Ref. 1. Of course, another critical equilibrium condition is $\tan\theta \le \mu_s$, where μ_s is the static friction coefficient, ensuring the incline friction can balance the component of gravitational force along the incline, the same as a block resting on a rough incline.

One can check the stability of the two equilibriums. Suppose the system is initially at A, as that shown in Fig. 1. Now the disk is made to rotate slightly clockwise (counterclockwise), m rotates and then CM will go to the RHS (LHS) of the vertical line at O, hence the then torque due to W about O will make a further clockwise (counterclockwise) rotation, meaning CM will go further away from A. In other words, the equilibrium at A is unstable. By the same method, the equilibrium at B is found to be stable. Besides, those who wish to understand more about the stability properties of the system could refer back to Ref. 1 and the related paper by De Luca.[3]

Finally, I ought to say that the analytical approach in Ref. 1 is worthy nevertheless, since one can learn and appreciate the reasoning in every step of the process of solving the problem by employing simple concepts in trigonometry and geometry. Indeed, an alternative "simpler approach" of a physics problem is very often unavailable.

1. Roberto De Luca, "Will the wheel stand still uphill?" *Phys. Teach.* **59**, 430–431 (Sept. 2021).
2. See, for example, http://hyperphysics.phy-astr.gsu.edu/hbase/cm.html.
3. R. De Luca, "Oscillation of a balanced hollow cylinder on an inclined plane," *Am. J. Phys.* **89**, 677–682 (July 2021).

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